The Quantum Hall Effects

Integer and Fractional

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Outline

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Experimental Setup

- Temperature: $< 5\,\mathrm{K}$ (He cyrostat)
- Small B (some Tesla)
- Measure ρ_{xx} and ρ_{xy} vs. B

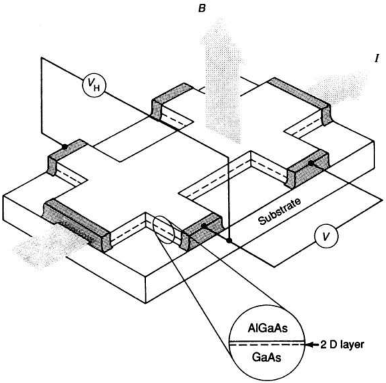

Figure: Hall bar
Eisenstein *et al.* *Science* **248**, 1510 (1990)

Two Dimensional Electron Gas

- QHE based on almost perfect realization of 2DEG

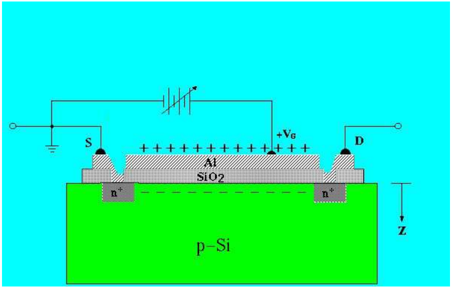

Figure: Si MOSFET

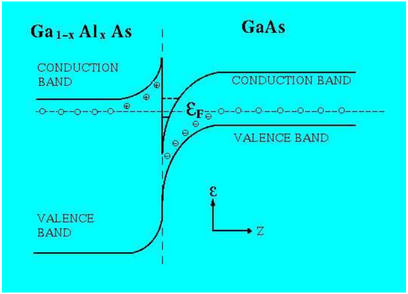

Figure: Energy level diagram

Ming Qiu – Johns Hopkins University: pha.jhu.edu/

QHE – Experimental Results

- $\rho_H = \rho_{xy} = -\frac{1}{n}\frac{h}{e^2}$ with $n = 1, 2, 3, \dots$ with precision 10^{-10}
- At plateaus $\rho_{xx} \to 0$
- Independent of material parameters

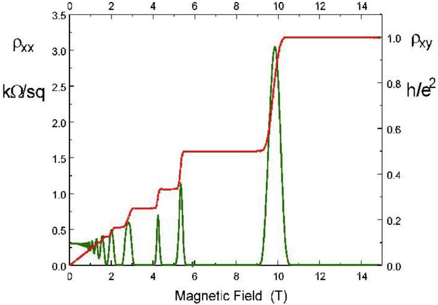

Figure: ρ_{xx} and ρ_{xy}

en.wikipedia.org

Electrons in a Strong Magnetic Field

- Magnetic length: $\ell_0 = \sqrt{\hbar c / eB}$
- Cyclotron frequency: $\omega_c = eB/m$
- Landau levels: $E_n = (n + \frac{1}{2})\hbar\omega_c$ with $n = 0, 1, 2 \dots$
- Degeneracy of Landau levels: $N_S = \frac{e}{hc}\mathbf{\Phi} = \frac{\mathbf{\Phi}}{\mathbf{\Phi}_0}$
- Filling factor $n = 2\pi\ell_0^2 n_0 = n_0/n_B$

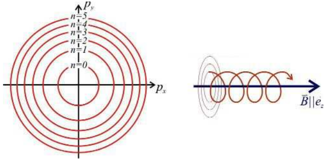

Figure: Landau levels

de.wikipedia.org

2D Density of States

- Due to impurities degeneracy of states with different (X, Y) lifted $\Rightarrow$ Extended and localized states

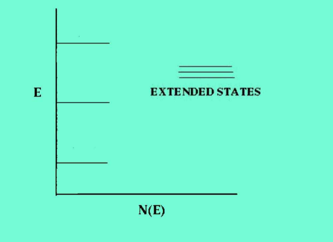

Figure: DOS without impurities

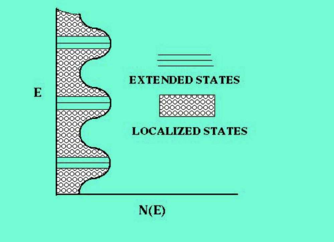

Figure: DOS with $U(\mathbf{r})$

Qui: John Hopkins University

Quantization of Conductance

Since $n \in \mathbb{N} \Rightarrow n_0 = n n_B$ with $n \in \mathbb{N}$

$$\Rightarrow \sigma_H = -ecn_0/B = ecnn_B/B = \boxed{-ne^2/h}$$

Hall conductance

$$\sigma = \begin{bmatrix} 0 & ne^2/h \\ -ne^2/h & 0 \end{bmatrix}$$

Currents at the Edge

- Why exact quantization independent of material?

Hall conductance

Edge currents only contribute

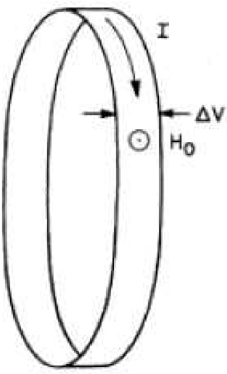

Figure: Hall ribbon Laughlin, *PRB* **23**, 5632 (1980)

Charge transport in the IQHE

Adding one flux quantum

$$x_k^m \longrightarrow x_k^{m-1}$$

Charge transport between edges

- States in Landau level shift over one by one
  $\longrightarrow$ Edge to edge transfer of one e^- per Landau level

$$I = -\frac{ne^2 V_H}{h}$$

Phenomenology of the FQHE

- $B \approx 15\,\mathrm{T}$:
- $T < 1$ K and pure samples required

Fractional n

Hall conductance exhibits plateaus at at $n = 4/3,\ 5/3,\ 7/3,\ 1/5$ and so on

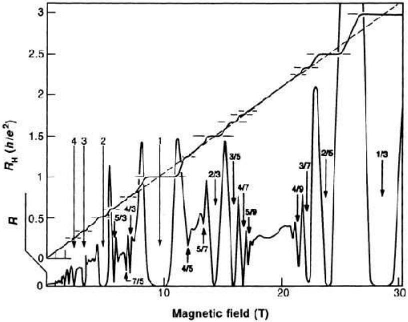

Figure: Hall resistance R_H and dissipative resistance R
Laughlin, *PRB* **23**, 5632 (1980)

Laughlin Wave Functions

- Choose symmetric gauge → rotational symmetry about origin preserved
- Angular momentum m good quantum number
- Consider only lowest Landau level:

$$\Psi(x,y) = f(z)e^{-\frac{1}{4}|z|^2}$$

- Choose: $f(z) = \prod_{j=1}^{N}(z - Z_j)$

Laughlin Wave Function

- Effect of Coulomb interaction non-trivial
- Interacting electron model required
- Two-body problem for particles:

$$\Psi_{mM}(z_1, z_2) = (z_1 - z_2)^m (z_1 + z_2)^M e^{-\frac{1}{4}\left(|z_1|^2 + |z_2|^2\right)}$$

- For fermions m must be odd
- Exact solution for general $V\left(|\mathbf{z}_1 - \mathbf{z}_2|\right)$
- Only one state with fixed m and M with energy eigenvalue:

$$\text{Haldane pseudopotential: } v_m = \frac{\langle mM|V|mM\rangle}{\langle mM|mM\rangle}$$

Many-body states

- Without magnetic field only continous spectrum

Bound states

Discrete spectrum
→ Bound states exist

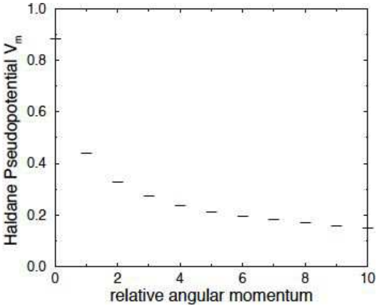

Figure: v_m vs. m in units of $e^2/\epsilon_0\ell$

Girvin *Séminaire Poincaré* **2** (2004) 53

Many-body states

- We need to find: $\Psi\left(\{z\}\right) = f\left(\{z\}\right) e^{-\frac{1}{4}\sum_j |z_j|^2}$
  - f polynomial: Slater determinant with all states occupied

$$\begin{aligned} f\left(\{z\}\right) &= \begin{vmatrix} (z_1)^0 & (z_2)^0 \\ (z_1)^1 & (z_2)^1 \end{vmatrix} = (z_1)^0(z_2)^1 - (z_2)^0(z_1)^1 \\ &= (z_2 - z_1) \end{aligned}$$

  - Single Slater determinant to fill first N m-states:

$$f_N\left(\{z\}\right) = \prod_{i<j}^{N}(z_i - z_j)$$

- Thus full probability distribution:

$$\boxed{\left|\Psi\left(\{z\}\right)\right|^2 = \prod_{i<j}^{N} |z_i - z_j|^2 e^{-\frac{1}{2}\sum_{j=1}^{N} |z_j|^2}}$$

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The Plasma Physics Analogy

Consider Boltzmann weight $|\Psi(\{z\})|^2 = \exp(-\beta U_{\text{class}})$

Potential energy of uniform electron gas

$$\underbrace{U_{\text{class}}}_{\text{Jellium}} = \underbrace{m^2 \sum_{i<j} \left(-\ln|z_i - z_j|\right)}_{\text{interaction of particles with charge } m} + \underbrace{\frac{m}{4} \sum_k |z_k|^2}_{\text{const. charge density}}$$

- Electrostatics: $\Phi(\mathbf{r}) = Q\left(-\ln \frac{r}{r_0}\right)$
- Poisson equation $\Delta(1/4)|z|^2 = -(1/\ell^2) = 2\pi\rho_B \Rightarrow \rho_B = -1/(2\pi\ell^2)$

Plasma neutrality condition

$$nm + \rho_B = 0 \Rightarrow n = \boxed{\frac{1}{m}\frac{1}{2\pi\ell^2}}$$

Adding One Flux Quantum

- Adding one flux quantum:

$$\begin{aligned}\Psi_m^+(z_0; z_1, \cdots, z_N) &= \prod_{j=1}^{N}(z_j - z_0)\Psi_m(z_1, \cdots, z_N) \\ &= A_{z_0}^+\Psi_m(z_1, \cdots, z_N)\end{aligned}$$

- Energy of many-body state with “quasi-hole”:

$$\Phi_{pq}(z_0; z_1, \cdots, z_N) = \Phi(z_1, \cdots, z_N) - \frac{2}{m}\sum_{j=1}^{N} \ln |z_j - z_0|$$

- Energy to destroy or create particle at $B = 15\,\mathrm{T}$: $4\,\mathrm{K}$ → gap to all excitations
- Excitation energy: Energy for adding quasi particle with $Q = \pm e/m$

Illustration of the Wave Function

Wave function for completely filled Landau level:

$$\Psi_m(\{\mathbf{r}\}) = \prod_{1 \leq j < k \leq N} (z_j - z_k)^{m-1} \Psi_1(\{\mathbf{r}\})$$

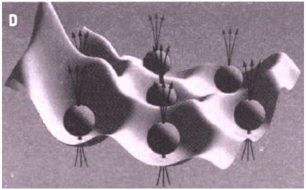

Figure: Representation of $n = 1/3$ state Eisenstein *et al.*, *Science* **248**, 1510 (1990)

Completely filled Landau level with each e^- having $m-1$ flux quanta attached

Summary

- IQHE arises from boundary conditions and is carried by edge states
  - Disorder essential
  - Non-interacting electrons
- FQHE electrons in lowest Landau level condense into Laughlin liquid with fractional excitations having a gap
  - Pure samples needed
  - Interacting electrons

Literature

- *Advanced Solid State Physics*
  Westview Press
- *The Quantum Hall Effects – Fractional and Integral*
  Springer
- *Science* **248**, 1510 (1990)
- *Phys. Rev. B* **23**, 5632 (1980)
- *Séminaire Poincaré* **2** (2004) 53